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# A DYNAMIC GREEN FUNCTION FORMULATION FOR THE RESPONSE OF A BEAM STRUCTURE TO A MOVING MASS

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A dynamic Green function approach is used to determine the response of a simply supported Bernoulli–Euler beam of finite length subject to a moving mass traversing its span. The proposed method produces a simple matrix expression for the deflection of the beam. The efficiency and simplicity of the method is demonstrated by several numerical examples. The effect of various parameters on the dynamic response is investigated.



## 1. INTRODUCTION

The moving load problem is a fundamental problem in structural dynamics. The importance of this problem is manifested in numerous applications in the field of transportation. Bridges, guideways, overhead cranes, cableways, rails, roadways, runways, tunnels and pipelines are example of structural elements to be designed to support moving masses. Also, in connection with the design of machining processes, many members can be modelled as beams acted upon by moving loads. The challenge of these designs has attracted the attention of many investigators since 1897, when the Chester Rail Bridge collapsed in England [1]. Various kinds of problems associated with moving loads have been presented in the excellent monograph by Frýba [2]. More recent developments and results can be found in state-of-the-art reviews [3–8].

The first dynamic analyses of structures under moving loads involved a simply supported beam in two limiting cases. In the first case, the effects of the load inertia were neglected in the analysis, and the problem is commonly called a moving force approximation. The methods of solution were generally within the framework of modal expansion and linear transformation techniques. A closed form solution can be obtained when the force travels with a constant velocity. This classical case was first solved by Krylov [9] and then by Timoshenko [10]. In the second case, the inertial effects of the beam were neglected. This problem was originally formulated and approximately solved in the first half of the nineteenth century by Zimmermann [11] and Stokes [12].

The above analyses introduced extreme approximations in modelling the physical problem of moving loads. Modern computational techniques have elminated the needs for many of the limitations and simplifications previously imposed. As a consequence, more precise modelling is required for the simulation of physical systems.

The double Laplace transform has been utilized by Hamada [13] to find a solution for a beam with damping under the action of a moving massless load. Olsson [14] presented analytical and finite element solutions of a simply supported beam subjected to a constant

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force moving at constant speed. His analytical solution is a series solution, as was given in reference [2]. Mackertich [15] studied the response of a simply supported beam excited by a moving force based on the Timoshenko beam theory. In another recent paper, Mackertich [16] used the modal superposition method for the beam deflection and compared the response of a Timoshenko beam to a moving mass to that of a Bernoulli–Euler one. He approximated the total time derivative of the mass displacement by the partial derivative to circumvent the difficulty that arises from the existence of a coupling term in the mass acceleration expression. A very recent work by Michaltsos *et al*. [17] follows the same approximation to derive a series solution for beam dynamic deflection in terms of beam normal modes, by using as a first approximation the solution of the corresponding problem without the effect of the mass inertia. More recently, using the well known assumed mode method, Lee [18] presented a numerical solution based on integration programs using the Runge–Kutta method for integrating the response of clamped–clamped beam acted upon by a moving mass. He pointed out the possibility of the mass separating from the beam during the course of motion by monitoring the contact force between the mass and the beam.

Ting *et al*. [19] formulated and solved the problem using the influence coefficients (static Green function). The distributed inertial effects of the beam were considered as applied external forces. Correspondingly, at each position of the mass, numerical integration had to be performed over the length of the beam. However, the use of the dynamic Green function in the present investigation makes the deflection expression for the beam to be written in a simple form. This is an auxiliary benefit of the current investigation over the earlier work, because the computation becomes more efficient. We recall that Ting *et al*. used influence coefficients, which are the static deflection due to a unit force.

An exact and direct modelling technique is presented in this paper for modelling beam structures subjected to a mass moving at constant speed. This technique is based on the dynamic Green function. The equation of motion in matrix form is formulated and is non-dimensionalized so that the numerical results presented are applicable to large combinations of system parameters. The computational algorithm presented is straightforward and efficient. In order to demonstrate the procedure and to show the simplicity and efficiency of the method presented, quantitative examples are given. The influence of variation of the parameters of the system on the dynamic response is studied. In addition, the physical implications of the mathematical results are addressed.

## 2. PROBLEM FORMULATION

The differential equation of a Bernoulli–Euler beam of finite length, subject to a concentrated force, is

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + m\frac{\partial^2 w(x,t)}{\partial t^2} = F\delta(x-u), \tag{1}$$

where $E$ is Young's modulus, $I$ is the second moment of area of the beam cross-section, $m$ is the mass per unit length, $x$ is the axial co-ordinate, $t$ is the time, $w(x, t)$ is the transverse deflection of the beam, $F$ is the applied force and $\delta(x-u)$ is the Dirac delta function. The boundary conditions for a simply supported beam and the initial conditions are

$$w(x,t) = \frac{\partial^2 w(x,t)}{\partial x^2} = 0 \qquad \text{at } x = 0 \text{ and } L, \tag{2}$$

$$w(x,0) = \frac{\partial w(x,0)}{\partial t} = 0. \tag{3}$$

Referring to Figure 1, $F$ is the reaction force exerted by the mass $M$ on the beam. For the mass shown in the free body diagram, Newton's second Law yields

$$F = M\left(g - \frac{\mathrm{d}^2\beta}{\mathrm{d}t^2}\right), \tag{4}$$

where $g$ is the gravitational acceleration and $\beta$ is the transverse displacement of the mass. Therefore, in the above and hereinafter we use the notation

$$\beta(t) = w(x, t)|_{x=u}. \tag{5}$$

The dynamic Green function is utilized to find the solution for the stated problem. Hence, if $G(x, u)$ is the dynamic Green function, as yet unknown, for the stated problem, then the solution of equation (1) takes the form

$$w(x, t) = G(x, u)F, \tag{6}$$

where $G(x, u)$ is the solution of the differential equation

$$\frac{\mathrm{d}^4W(x)}{\mathrm{d}x^4} - q^4w(x) = \delta(x-u), \tag{7}$$

where $q$ is the frequency parameter (separation constant) and is given by

$$q^4 = \omega^2 m/EI, \tag{8}$$

in which $\omega$ is the circular frequency that expresses the motion of the mass and is equal to $\pi v/L$.

The solution of equation (7) is assumed in the form [20]

$$G(x, u) = \begin{cases} A_1\cos(qx) + A_2\sin(qx) + A_3\cosh(qx) + A_4\sinh(qx), & 0 \leqslant x \leqslant u, \\ B_1\cos(qx) + B_2\sin(qx) + B_3\cosh(qx) + B_4\sinh(qx), & x \leqslant u \leqslant L, \end{cases} \tag{9}$$

The eight constants $A_1, \ldots, A_4$ and $B_1, \ldots, B_4$ are evaluated such that the Green function $G(x, u)$ satisfies the following conditions [21]: (a) two boundary conditions at each end of the beam depending on the type of end support—for a simply supported beam,

$$G(0, u) = G(L, u) = G''(0, u) = G''(L, u), \tag{10}$$

where the prime indicates a derivative with respect to $x$; (b) continuity conditions of displacement, slope and moment at $x = u$, i.e.,

$$G(u^+, u) = G(u^-, u), \qquad G'(u^+, u) = G'(u^-, u), \qquad G''(u^+, u) = G''(u^-, u); \tag{11}$$

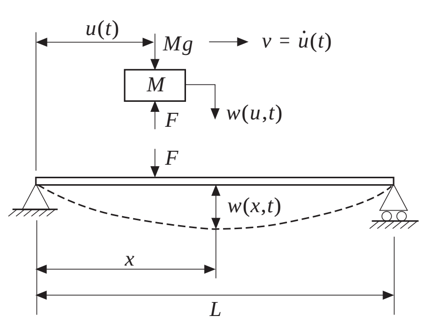


Figure 1. A mass traversing a beam with constant velocity.

(c) shear force discontinuity of magnitude one at $x = u$, i.e.,

$$EI[G'''(u^+, u) - G'''(u^-, u)] = 1. \tag{12}$$

The Green function as determined by the above procedure is given by

$$G(x, u) = \frac{1}{2EIq^3 \sin(qL) \sinh(qL)} \begin{cases} g(x, u), & 0 \leqslant x \leqslant u, \\ g(u, x), & x \leqslant u \leqslant L, \end{cases} \tag{13a}$$

where

$$g(x, u) = \sinh(qL) \sin(qx) \sin(qL - qu) - \sin(qL) \sinh(qx) \sinh(qL - qu), \tag{13b}$$

and $g(u, x)$ is obtained by switching $x$ and $u$ in $g(x, u)$. This follows from the fact that $G(x, u)$ must be symmetric to satisfy the Maxwell–Rayleigh reciprocity law.

It is to be noticed that when $q$ is equal to zero, the expression given by equation (13) reduces to the static Green function (beam influence coefficients). Specifically,

$$\lim_{q \to 0} G(x, u) = \frac{L^3}{6EI}\left(1 - \frac{u}{L}\right)\frac{x}{L}\left[2 - \left(\frac{x}{L}\right)^2 - \left(\frac{u}{L}\right)^2\right], \qquad 0 \leqslant x \leqslant u. \tag{14}$$

It proves convenient [19] to change the variable by using the relationship $u = u(t)$, so that

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} = \frac{\mathrm{d}\beta(t)}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}t} = v\frac{\mathrm{d}\beta(u)}{\mathrm{d}u},$$

$$\frac{\mathrm{d}^2\beta}{\mathrm{d}t^2} = v^2\frac{\mathrm{d}^2\beta(u)}{\mathrm{d}u^2}, \tag{15}$$

where

$$\beta(u) = w(x, u)|_{x=u}. \tag{16}$$

Eliminating $F$ between equations (4) and (6) and making use of equation (15) yields

$$w(x, u) = G(x, u)M\left[g - v^2\frac{\mathrm{d}^2\beta(u)}{\mathrm{d}u^2}\right], \tag{17}$$

Equation (17) is a second order differential equation, which specifies the beam deflection at position $x$ caused by the load at location $u$. It represents the particular solution for the governing equation (equation (1)); i.e., the forced vibration part of the deflection. The boundary conditions, equations (2), are embedded in the Green function. However, one still needs to satisfy the two initial conditions given by equation (3). Therefore, one should add the complementary solution $w_d(x, u)$, which is given by [13]:

$$w_d(x, u) = \frac{-2MgL^3}{\pi^4 EI}\sum_{j=1}^{\infty}\frac{\alpha}{j^3(j^2 - \alpha^2)}\sin\left(\frac{j\pi x}{L}\right)\sin\left(\frac{j^2\pi u}{\alpha L}\right), \qquad 0 \leqslant x, u \leqslant L, \tag{18a}$$

where the speed parameter $\alpha$ is defined as

$$\alpha = \frac{vL}{\pi}\sqrt{\frac{m}{EI}}. \tag{18b}$$

It is to be noted that for a moving force problem, in which the inertial term of the moving mass is removed, the forced vibration part of the deflection as is given by equation (17) is

$$w_f(x, u) = \frac{Mg}{2EIq^3 \sin(qL) \sinh(qL)} \begin{cases} g(x, u), & 0 \leqslant x \leqslant u, \\ g(u, x), & x \leqslant u \leqslant L. \end{cases} \tag{19}$$

According to formula (1.31) in reference [2], the well known Fourier series solution of the moving force at a non-critical speed is

$$W(x, u) = W_d(x, u) + w_f(x, u), \qquad 0 \leqslant x, u \leqslant L, \tag{20a}$$

where

$$w_f(x, u) = \frac{2MgL^3}{\pi^4 EI} \sum_{j=1}^{\infty} \frac{1}{j^2(j^2 - \alpha^2)} \sin\left(\frac{j\pi x}{L}\right) \sin\left(\frac{j\pi u}{L}\right), \qquad 0 \leqslant x, u \leqslant L, \tag{20b}$$

and $w_d(x, u)$ is given by equation (18a).

Thus, by making use of the dynamic Green function, the sum of the Fourier series, equation (20b), has been obtained in a closed form.

It is also interesting to note that the expression given by equation (19) is exactly the forced vibration part obtained by Hamada [13], using the double Laplace transform approach (second part of equation (7) in reference [13]). Specifically,

$$w_f(x, u) = \frac{2MgL^3}{\pi^4 EI} \left[ \frac{\sin(\alpha\pi u/L)}{\sin \alpha\pi} \sin \alpha\pi\left(1 - \frac{x}{L}\right) - \frac{\alpha\pi u}{L}\left(1 - \frac{x}{L}\right) \right.$$

$$\left. - \sin \alpha\pi\left(\frac{u}{L} - \frac{x}{L}\right) + \alpha\pi\left(\frac{u}{L} - \frac{x}{L}\right) \right], \qquad x \leqslant u \leqslant L, \tag{21}$$

Therefore, equations (19), (20b) and (21) are different representations for the forced vibration part of the deflection. Verification of this result may be obtained by evaluating these three expressions numerically, after recalling the fact that $q = \pi(\alpha)^{1/2}$ and truncating the series solution given by equation (20b) after 12 terms. This is shown in Figure 2 for different speed parameters $\alpha$. The three representations are identical up to the fourth decimal place. However, one needs elegant procedures to prove mathematically that these three expressions are equivalent.

## 3. COMPUTATIONAL ALGORITHM

Since the closed form solution of equation (17) is not known, one seeks an approximate solution in which one replaces the derivatives by its finite difference approximation. Hence the beam is divided into $(N - 1)$ intervals of length $h$. The discretized version of equation (17) is

$$w(x_i, u) = G(x_i, u)M\left[g - v^2 \frac{\mathrm{d}^2\beta(u)}{\mathrm{d}u^2}\right], \tag{22}$$

where the subscript $i$ refers to any one of the $N$ discrete station points. It is to be noted that equation (22) is explicit in $u$ and, accordingly, implicit in time $t$. Without loss of generality, the variable $u$ that represents the location of the mass is discretized in the same

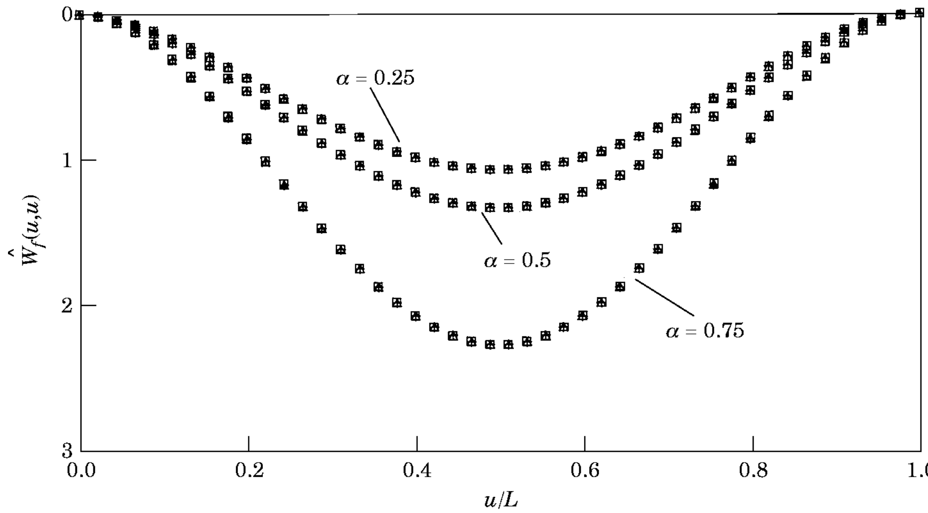


Figure 2. A comparison of different representations for the forced vibration part of the deflection when $\alpha = 0{\cdot}25$, $0{\cdot}5$ and $0{\cdot}75$. □ □ □ □, equation (19); △ △ △ △, equation (20b); + + + +, equation (21).

way. Therefore the increments of $u$ are also taken to be of length $h$. This incrementation is equivalent to using time intervals that are the length of the time required to travel from any one of the $N$ stations to the next adjacent one.

All that remains to be done is to use finite divided difference formulas to represent the $u$ derivative in equation (22). The Houbolt method [22] is chosen, since $u$ is equivalent to the time variable. Therefore, upon letting $f(u)$ be any sufficiently smooth function, the approximate formula is [22]

$$\left.\frac{\mathrm{d}^2 f}{\mathrm{d}u^2}\right|_{u_j} = \frac{1}{h^2} \sum_{k=0}^{3} a_k f(u_{j-k}) + O(h^2), \tag{23}$$

where $u_j$ indicates that the mass is at the $j$th station point. The coefficients $a_k$ are $a_0 = 2$, $a_1 = -5$, $a_2 = 4$ and $a_3 = -1$.

Application of equation (23) to equation (22) results in the following set of algebraic equations:

$$h^2 w(x_i, u_j) = G(x_i\, u_j) M \left[ h^2 g - v^2 \sum_{k=0}^{3} a_k\, w(x_{j-k}, u_j) \right], \qquad i, j = 1, 2, \ldots, N, \tag{24}$$

where use has been made of equation (16) with the remark that

$$\beta(u_j) = w(x, u_j)|_{x=u_j} = w(x_j, u_j). \tag{25}$$

Equation (24) is the same as equation (11) in reference [19], neglecting the numerical integration in that equation, replacing the influence coefficient by the dynamic Green function, and setting $\dot{v} = 0$.

Equation (24) can be expressed concisely in matrix form:

$$[h^2[\mathbf{I}] + a_0\, v^2 [\mathbf{G}] M [\mathbf{P}_{ij}]]\{\mathbf{w}_{u_j}\} = M[\mathbf{G}] \left[ h^2 g \{\boldsymbol{\Delta}\} - v^2 \sum_{k=1}^{3} a_k\, [\mathbf{P}_{j,j-k}]\{\mathbf{w}_{u_{j-k}}\} \right], \tag{26}$$

where $[\mathbf{I}]$ is the identity matrix and

$$\{\mathbf{w}_{u_j}\} = \{w(x_1, u_j), w(x_2, u_j), \ldots, w(x_N, u_j)\}^{\mathrm{T}},$$

$$\{\boldsymbol{\Delta}_{u_j}\} = \{\underset{\substack{\uparrow \\ j\text{th column}}}{0 \cdots 010 \cdots 0}\}^{\mathrm{T}},$$

$$[\mathbf{G}] = \begin{bmatrix} G(x_1, u_1) & \cdots & G(x_1, u_N) \\ \cdots & & \cdots \\ \cdots & & \cdots \\ \cdots & & \cdots \\ G(x_N, u_1) & \cdots & G(x_N, u_N) \end{bmatrix}, \qquad [\mathbf{P}_{i, j-k}] = \begin{array}{c} \overset{(j-k)\text{th column}}{\downarrow} \\ \begin{bmatrix} 0 \cdots\cdots\cdots 0 \\ \cdots\cdots\cdots\cdots \\ 0 \cdots 0\ 1\ 0 \cdots 0 \\ 0 \cdots\cdots\cdots 0 \\ 0 \cdots\cdots\cdots 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \leftarrow j\text{th row.} \\ \\ \\ \end{array} \tag{27}$$

Equation (26) describes the transverse displacement of the beam when the mass is at any station $j$. In order to include the initial conditions, equations (3), which describes the displacement and slope of the beam at onset of motion, one refers back to the change of variables introduced earlier. Accordingly, the initial conditions correspond to the mass being at the initial station point $j = 0$. For this station, equation (26) requires values of $\{\mathbf{w}_{u_k}\}$, where $k \leqslant 0$. These values may be set equal to zero.

Equation (26) may be placed into non-dimensional form so that the numerical results presented are applicable for large combinations of system parameters. This is achieved by letting $w_{st}$ be the scaling factor for the transverse displacement, where $w_{st}$ is the static deflection at the beam mid-span due to the weight of the mass, and by letting $T$ be the time scale, where $T$ is the period of the lowest vibration mode of the beam. Thus

$$w_{st} = \frac{MgL^3}{48EI}, \qquad T = \frac{2L^2}{\pi}\sqrt{\frac{m}{EI}}. \tag{28}$$

The appropriate non-dimensional quantities can be

$$\hat{w} = w/w_{st}, \qquad \hat{G} = \frac{EI}{L^3}G. \tag{29}$$

Upon using equation (29), the resulting non-dimensional form of equation (26) is found to be

$$[[\mathbf{I}] + \gamma a_0 [\hat{\mathbf{G}}][\mathbf{P}_{j,}]]\{\hat{\mathbf{w}}_{u_j}\} = [\hat{\mathbf{G}}]\left[48\{\boldsymbol{\Delta}\} - \gamma \sum_{k=1}^{3} a_k [\mathbf{P}_{j,j-k}]\{\hat{\mathbf{w}}_{u_{j-k}}\}\right], \tag{30}$$

where the non-dimensional parameter $\gamma$ depends on the mass ratio $M/mL$, the speed ratio, as given by equation (18b), $\alpha = v/v_{cr}$, and the number of segments, and is given by

$$\gamma = \pi^2 \left(\frac{M}{mL}\right)\left(\frac{L}{h}\right)^2\left(\frac{v}{v_{cr}}\right)^2. \tag{31}$$

where the critical speed $v_{cr}$ is defined as

$$v_{cr} = 2L/T \equiv \frac{\pi}{L}\sqrt{\frac{EI}{m}}. \tag{32}$$

The case $v/v_{cr} = 1$ corresponds to resonance with the fundamental mode when the load is a constant force.

## 4. EXAMPLES

The parameters selected for the given examples correspond to the data used in reference [17]. From equation (30) one can see that $\hat{w}(x, u)$ depends on only three non-dimensional parameters, $x/L$, $u/L$ and $\gamma$. Quantities such as mid-span dynamic magnification factors $\hat{w}(x/2, u/2)$ are thus functions of $\gamma$ alone. Consequently, the parameter $\gamma$ plays an important role in understanding the dynamic characteristics of the fundamental moving mass problem. For a moving force without mass, the response $\hat{w}(x, u)$ depends on only $x$, $u$ and $\alpha$. In Figure 3 is shown the influence of variation of the speed parameter $\alpha$ on the mid-span deflection for a moving force (i.e., the inertia of the moving mass is neglected). As $\alpha$ increases one can see that histories or influence lines deviate more from the quasi-static ones ($\alpha = 0$). One can also note that the curves for $\alpha = 0{\cdot}05$, $0{\cdot}125$ and $0{\cdot}5$ oscillate about the quasi-static influence lines. For these values of $\alpha$, the travel times ($L/v$) are ten, four and two times larger than the lowest vibration period, respectively. Hence, the lowest vibrating beam mode has enough time to complete ten, four and two vibration cycles, respectively. From the curve for $\alpha = 0{\cdot}25$, the maximum deflection $\hat{w} = 1{\cdot}2497$ occurs when the load is located at $u/L = 0{\cdot}4$, while for $\alpha = 0{\cdot}375$, the maximum deflection becomes $\hat{w} = 1{\cdot}5608$, taking place when the load is located at $u/L = 0{\cdot}5444$. In Figure 4, the inertia of the moving mass is included. The mass ratio $M/mL$ is $0{\cdot}3$. For $\alpha = 0{\cdot}25$, the maximum deflection is $\hat{w} = 1{\cdot}3496$ when the mass is located at $u/L = 0{\cdot}4222$, while for $\alpha = 0{\cdot}375$ the maximum deflection is $\hat{w} = 1{\cdot}6051$ when the mass is at $u/L = 0{\cdot}5556$.

In Figure 5, the mass ratio $M/mL = 0{\cdot}6$. For $\alpha = 0{\cdot}25$, the maximum deflection $\hat{w} = 1{\cdot}2531$ occurs when the mass is located at $u/L = 0{\cdot}3889$, while for $\alpha = 0{\cdot}375$ the maximum deflection is $\hat{w} = 2{\cdot}056$, which occurs when the mass is at $u/L = 0{\cdot}5444$. It can

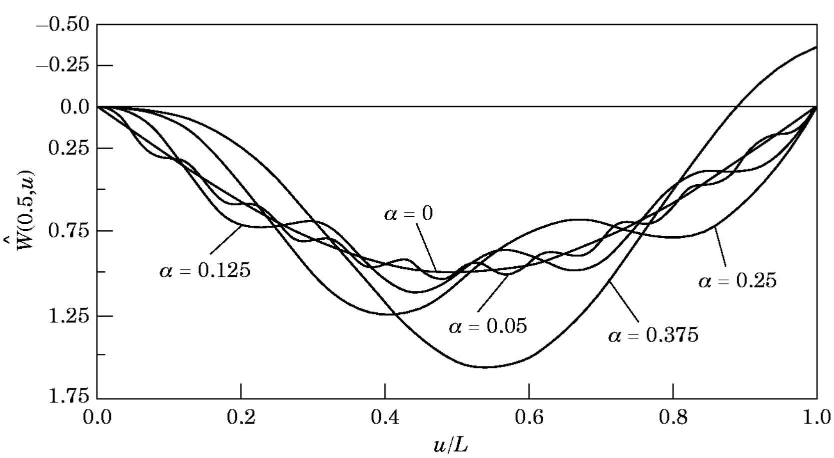


Figure 3. The central deflection of a beam traversed by a moving force when $\alpha = 0$, $0{\cdot}05$, $0{\cdot}125$, $0{\cdot}25$ and $0{\cdot}375$.

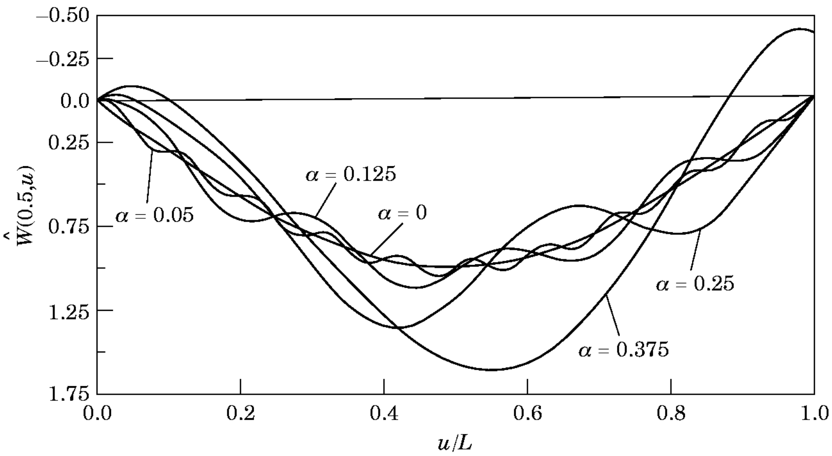


Figure 4. The central deflection of a beam traversed by a moving mass when $M/mL = 0{\cdot}3$; $\alpha = 0$, 0·05, 0·125, 0·25 and 0·375.

be concluded that if the moving mass is small in comparison to the mass of the beam, it is sufficient to consider the mass as a moving force only.

In Figures 6 and 7, the parameters selected correspond to data used in reference [16], which are for a typical simply supported beam representing a bridge with span $L = 50$ m, modulus of elasticity $E = 3{\cdot}34 \times 10^{10}$ N/m$^2$, moment of area of cross-section $I = 1{\cdot}042$ m$^4$ and mass per unit length $m = 4800$ kg/m, excited by a moving mass of $M = 50\,000$ kg. These data correspond to a mass ratio of $M/mL = 0{\cdot}2083$ and a critical velocity of $v_{cr} = 169{\cdot}2$ m/s. In Figure 6 is shown the deflection at mid-span of the beam when the effect of the load mass on the dynamic response is neglected for travelling velocities of $v = 25$ m/s, $v = 50$ m/s and $v = 100$ m/s, which correspond to $\alpha = 0{\cdot}1478$, $\alpha = 0{\cdot}2955$ and $\alpha = 0{\cdot}5911$, respectively. In Figure 7, the effect of the load mass is included. Comparing these two figures, one can conclude that the effect of the mass inertia is significant in determining the dynamic response of a beam excited by a high velocity moving load. In other words, the inertial effect of the moving mass cannot be neglected in comparison with the gravitational effect if the travelling velocity of the mass is not small.

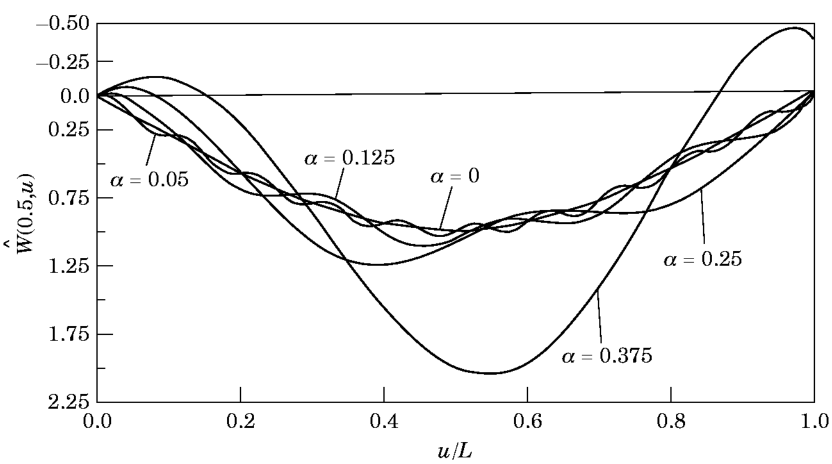


Figure 5. The central deflection of a beam traversed by a moving mass when $M/mL = 0{\cdot}6$; $\alpha = 0$, 0·05, 0·125, 0·25 and 0·375.

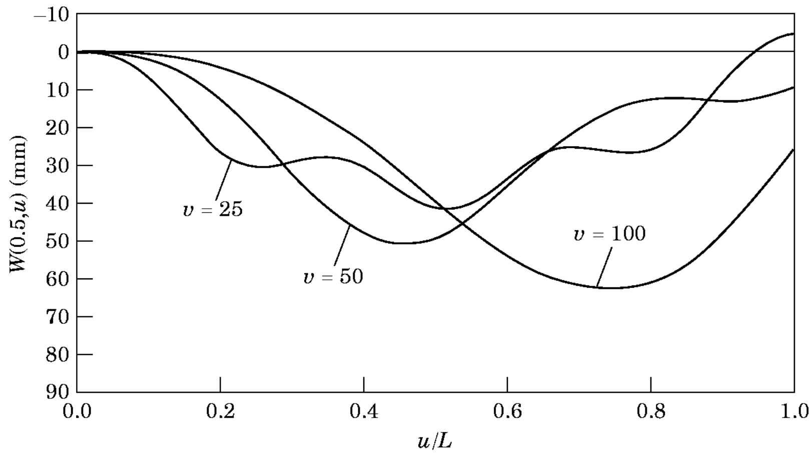


Figure 6. The central deflection of a beam traversed by a moving force when $v = 25$, 50 and 100 m/s.

The purpose of the foregoing presentation was to display the versatility of the algorithms developed. Beams with various combinations of boundary conditions and Timoshenko-type beams can be treated in a similar manner. Put simply, the Green function $G(x, u)$ in the algorithms presented is replaced by the proper one. In the Appendix, Green function for fixed–fixed beams and for cantilever beams are cited. Green functions for other complicated boundary conditions are tabulated in reference [20], while Green functions for Timoshenko-type beams are derived by Lueschen *et al*. [23].

## 5. CONCLUSIONS

The objective of this work was to present a simple and direct technique to treat the problem of a beam traversed by a moving mass. The influences of variations of the travelling velocity and the ratio of the moving mass to the mass of beam on the dynamic response are studied. Finally, the effect of neglecting the inertia of the mass on the dynamic response of the beam is demonstrated.

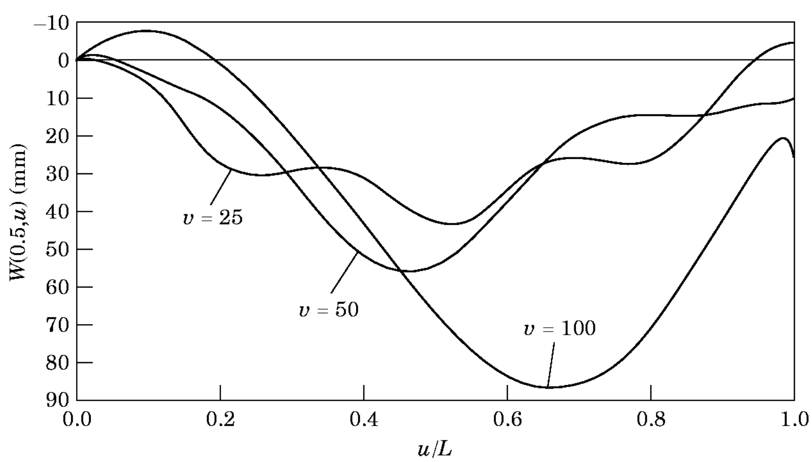


Figure 7. The central deflection of a beam traversed by a moving mass when $M/mL = 0{\cdot}2083$; $v = 25$, 50 and 100 m/s.

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## APPENDIX

The Green functions are given by

$$G(x, u) = \frac{1}{2EIq^3\Delta}\begin{cases} g(x, u), & 0 \leqslant x \leqslant u, \\ g(u, x), & x \leqslant u \leqslant L, \end{cases} \tag{A1}$$

where

$$g(x, u) = D_1 (\cos qx - \cosh qx) + D_2 (\sin qx - \sinh qx). \tag{A2}$$

For fixed–fixed beams,

$$\Delta = 2(1 - \cos qL \cosh qL), \tag{A3}$$

$$D_1 = (\cos qL - \cosh qL)(\sin z - \sinh z) - (\sin qL - \sinh qL)(\cos z - \cosh z), \tag{A4}$$

$$D_2 = (\sin qL + \sinh qL)(\sin z - \sinh z) + (\cos qL - \cosh qL)(\cos z - \cosh z). \tag{A5}$$

where $z = q(L - u)$, while for cantilever beams,

$$\Delta = 2(1 + \cos qL \cosh qL), \tag{A6}$$

$$D_1 = (\cos qL + \cosh qL)(\sin z + \sinh z) - (\sin qL + \sinh qL)(\cos z + \cosh z), \tag{A7}$$

$$D_2 = (\sin qL - \sinh qL)(\sin z + \sinh z) + (\cos qL + \cosh qL)(\cos z + \cosh z). \tag{A8}$$